

# Berry Phases, Quantum Phase Transitions and Chern Numbers

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## Abstract

We study the relation between Chern numbers and Quantum Phase Transitions (QPT) in the XY spin-chain model. By coupling the spin chain to a single spin, it is possible to study topological invariants associated to the coupling Hamiltonian. These invariants contain global information, in addition to the usual one (obtained by integrating the Berry connection around a closed loop). We compute these invariants (Chern numbers) and discuss their relation to QPT. In particular we show that Chern numbers can be used to label regions corresponding to different phases.

*Key words:* Berry phases, topological invariants, quantum phase transitions

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Recently, a close connection between Berry phases (BP) associated to quantum many-body systems and Quantum Phase Transitions (QPT) has emerged, attracting much attention [1,2,3,4,5]. From an experimental point of view such a connection is very interesting, due to the robustness of BP against continuous changes in the system's parameters. In particular, it has been proposed by Carollo et al. [1] that the existence of a QPT could be detected, without having to undergo a phase transition, through a cyclic evolution in parameter space. In that work, the authors have shown that a BP can be defined for the XY model whose behavior with respect to the system's parameters reflects the presence of QPT. They computed the relative geometric phase between the ground and first excited states for loops around the XX criticality region and found that its derivative presents a singular behavior at the critical point. Zhu [2], working also with the XY model, obtained similar results for the derivative of the BP corresponding to the ground state. He also analyzed the scaling behavior of the BP, showing that it can be used as a signature of quantum criticality. Plastina et al. [3] found, using the Dicke model, that the BP vanishes exactly (in the thermodynamic limit) in the region corresponding to the normal phase and is greater than zero in the super-radiant phase. They also found a singular behavior of the derivative of the BP at the

critical point. Yet another proposal to detect QPT using Berry phases has been put forward by Hama [4].

All these examples point towards a very general, model independent relation between BP and QPT. In order to understand the origin of this relation, it is necessary to recognize general patterns that may emerge from concrete examples. It is a well known fact, first pointed out by Simon [6] that given a parameter-dependent Hamiltonian  $H(\alpha)$ , the BP corresponding to the  $n^{th}$  (non-degenerate) energy band  $E_n(\alpha)$ , arises as the holonomy of a connection defined on the line bundle spanned by the family of all eigenvectors  $|\varphi_n(\alpha)\rangle$ , as  $\alpha$  varies over the parameter space, where

$$H(\alpha)|\varphi_n(\alpha)\rangle = E_n(\alpha)|\varphi_n(\alpha)\rangle. \quad (1)$$

But in the examples related to QPT studied so far, it is not clear what -exactly- the base space of the bundle is. In fact, all computations rely on the introduction of an additional parameter through a unitary transformation of the Hamiltonian. Then the Berry phase for a special class of loops in this extended parameter space is computed, leading to the results reported in references [1,2,3,4,5].

In this work, using the model proposed in [5], we intend to go a step further in the sense that, for each value of the external field ( $\lambda$ ), we identify a space (isomorphic to the two-sphere) and a corresponding line bundle  $L_\lambda$  over it, for which a topological invariant  $c_1(\lambda)$  can be computed. As shown below, this invariant is closely related to the phase transition of the XY model at the critical value  $\lambda = 1$ .

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Following Yuan et al. [5], we start by considering a system consisting of a spin chain coupled to a central spin, with Hamiltonian  $H = H_I + H_C + H_E$ , where

$$H_C = \frac{\mu}{2}\sigma^z + \frac{\nu}{2}\sigma^x, \quad (2)$$

$$H_E = -J \sum_{j=1}^N \left( \frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right), \quad (3)$$

$$H_I = \frac{Jg}{N} \sum_{j=1}^N \sigma_j^z \sigma_i^z. \quad (4)$$

The Pauli matrices  $\sigma^\alpha$  and  $\sigma_j^\alpha$  ( $\alpha = x, y, z$ ) describe, respectively, the central spin and the spin on the  $j^{\text{th}}$  site of the environmental spin chain. As can be read from eqns. (2)-(4), the parameters  $J, \lambda, g, \mu, \nu$  describe the coupling to the external field and the strength of the interaction among spins. The parameter  $\gamma$  accounts for the anisotropy in the spin chain. Assuming that the spin chain is in its ground state, the following effective mean-field Hamiltonian for the central spin can be obtained [5]:

$$H_{\text{eff}} = \left( \frac{\mu}{2} + \frac{2Jg}{N} \sum_{k=1}^{N/2} \cos \theta_k \right) \sigma^z + \frac{\nu}{2} \sigma^x. \quad (5)$$

Here  $\theta_k$  satisfies  $\cos \theta_k = (J\epsilon_k)/\sqrt{\epsilon_k^2 + \gamma^2 \sin^2(\frac{2\pi k}{N})}$ , with

$$\epsilon_k = \lambda - \cos\left(\frac{2\pi k}{N}\right) + \frac{g\mu}{N\sqrt{\mu^2 + \nu^2}}. \quad (6)$$

Following references [1,2,3,4,5], we change the Hamiltonian by means of a unitary transformation  $U(\varphi) = \exp(-i\varphi\sigma^z/2)$  to

$$\tilde{H} = U(\varphi)H_{\text{eff}}U^\dagger(\varphi). \quad (7)$$

If we keep all parameters appearing in  $\tilde{H}$  fixed, with the exception of  $\varphi$  and  $\gamma$ , we can regard this Hamiltonian as defined on the surface of a two dimensional sphere, obtained by stereographic projection from the plane with polar coordinates  $0 \leq \gamma \leq \infty$  and  $0 \leq \varphi < 2\pi$ , given that we restrict the parameter  $\nu$  to the limiting case  $\nu \ll 1$ . The compactification of the  $\varphi$ - $\gamma$  plane to a sphere is possible in that limiting case, because then the eigenvectors of  $\tilde{H}$  do not depend on  $\varphi$  when  $\gamma \rightarrow \infty$ . As mentioned above a parameter-dependent Hamiltonian induces, for each non-degenerate energy band, a line bundle over the parameter space. This line bundle comes equipped with a connection whose holonomy is precisely the geometric or Berry phase corresponding to the given eigenstate [6,7]. In the present case, we are regarding the Hamiltonian  $\tilde{H}$  as depending on the two parameters  $\varphi$  and  $\gamma$ . The ground state of  $\tilde{H}$  is readily shown [5] to be given by:

$$|g(\gamma, \varphi)\rangle = (\sin(\psi/2)e^{i\varphi}, -\cos(\psi/2)), \quad (8)$$

where

$$\sin \psi = \nu / \sqrt{\nu^2 + (\mu + 4Jg \sum_{k=1}^{N/2} \cos \theta_k / N)^2} \quad (9)$$

This ground state generates a line bundle  $L_\lambda$  over the sphere, whose topology is characterized by the first Chern number, a topological invariant that can be expressed as an integral over the parameter space (two-sphere) as [7]:

$$c_1(\lambda) = \frac{-i}{2\pi} \int P dP \wedge dP, \quad (10)$$

where  $P$  denotes the projector  $P = |g(\gamma, \varphi)\rangle\langle g(\gamma, \varphi)|$ .

After a long but straightforward calculation we obtain, in the thermodynamic limit,

$$c_1(\lambda) = \begin{cases} \frac{1}{2} (\text{sign}(\mu - 2Jg) - \text{sign}(\mu)), & \lambda < -1 \\ \frac{1}{2} (\text{sign}(\chi) - \text{sign}(\mu)), & \lambda \in [-1, 1] \\ \frac{1}{2} (\text{sign}(\mu + 2Jg) - \text{sign}(\mu)), & \lambda > 1, \end{cases} \quad (11)$$

with

$$\chi = \mu + \frac{4Jg}{\pi} \arcsin(\lambda). \quad (12)$$

Equation (11) is the main result of this paper. It shows that a topological invariant can be extracted from the effective Hamiltonian  $\tilde{H}$  that contains information about the QPT of the environmental spin chain at  $|\lambda| = 1$ . The interest of this result lies in the fact that it may be possible, in a general case, to express  $c_1(\lambda)$  as a function of the expectation values of certain physical observables. In contrast to BP, that depends not only on the topology of the state space but on the path followed in parameter space, the Chern number is a purely topological quantity, an integer that is robust against continuous perturbations of the Hamiltonian. This same idea can be applied to the ground state of the  $XY$  spin chain (without the coupling to a central spin). In that case, the Chern number scales with the number  $N$  of spins, but after normalization with  $N$ , one obtains a function that also reflects the presence of a QPT at the critical point. This result, and potential applications thereof, will be reported elsewhere.

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